NOISE-INDUCED OREDER : The Information Mixing(Theory of Dynamical Systems and Its Application to Nonlinear Problems)

AUTHOR(S):

Matsumoto, K.; Tsuda, I.

数理解析研究所講究録
536巻 1984年 190-200

NOISE-INDUCED ORDER

- The Information Mixing -

K. Matsumoto[1] and I. Tsuda[2]
松本健司、 津田一郎

1 Department of Physics, Kyoto University, Kyoto 606, Japan
2 Bioholonics Project, Research Development Corporation of Japan, Nissho bldg., 5F, 14-24 Koishikawa 4-chome, Bunkyo-ku, Tokyo 112, Japan

An information theory of one-dimensional mappings is constructed. The noise-induced order is further investigated in this framework. The curious phenomena about the mutual information are obtained.

1. INTRODUCTION

We have found a curious noise-effect in a certain class of one-dimensional mappings - Noise-induced order -[1]. In our previous papers[1)-3)] on this phenomenon, studies from several directions were attempted as will be shown below. In maps of the B-Z type we have studied in a series of our paper, the transition to the order from the chaos was observed. On the contrary, in maps of the logistic type, external noise induces the transition from the periodic behaviour to the chaotic behaviour. There appears a broadening of the invariant density and of the power spectrum and an increase in the Lyapounov number.

The B-Z map is expressed as follows.

$$x_{n+1} = (x_n - 0.125)^{1/3} + 0.50607357 \ \exp(-x_n) + b \qquad \text{for } x < 0.3,$$

$$x_{n+1} = 0.121205692 \ 10\, x_n \exp\left(-\frac{10}{3} x_n\right)^{19} + b \qquad \text{for } x > 0.3. \tag{1}$$

Our first observation was that the originally positive Lyapounov exponent changes to a negative one as the noise level is increased (See

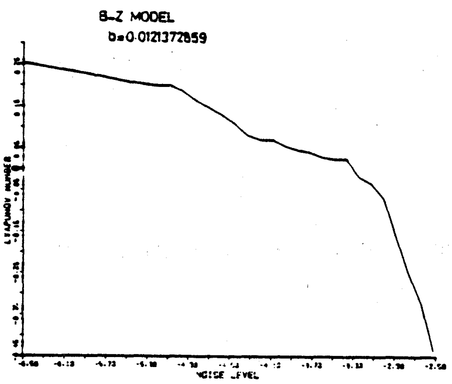

Fig. 1

Lyapounov exponent v,s, noise

Fig. 1). We studied the entropy of this system viewed as an information source, since the Lyapounov exponent is not an appropriate indicator in the presence of large noise. Regarding the R-L sequence of an orbit as the product of an information source, we calculated its entropy. As shown in Fig. 2, the entropy abruptly decreases as the noise level is increased. To check whether or not this phenomenon is observable in

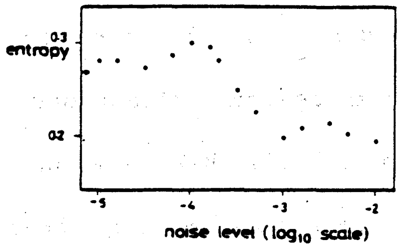

Fig. 2

Entropy v,s, noise

experiments, we studied the power spectrum. The results are shown in Fig. 3. Furthermore, we studied the feature of the orbit (equivalently, the invariant density) and observed the localization of orbits caused by the noise. This phenomenon indicates the difference of the character of the "randomness" between chaos and noise.

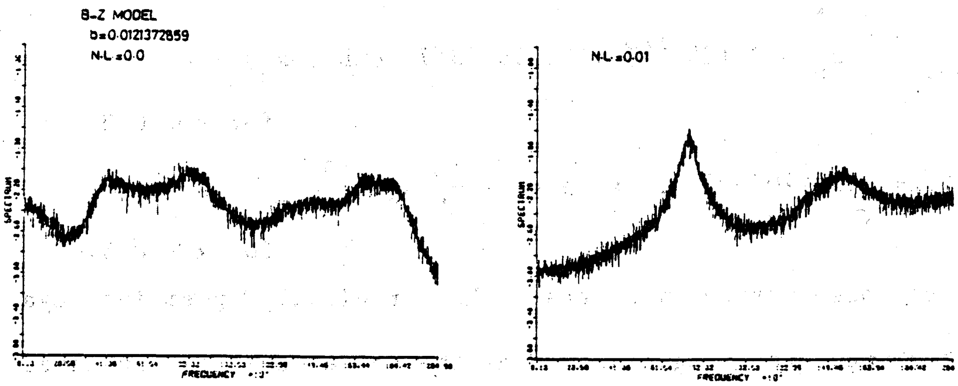

Fig. 3 Power Spectrum in the case without noise ((a)) and the case with noise ((b)).

Furthermore, this directly stems from a certain kind of nonuniformity character of the strange attractor, which produces the nonuniformity of the refinement of Markov partition (See Fig. 4).

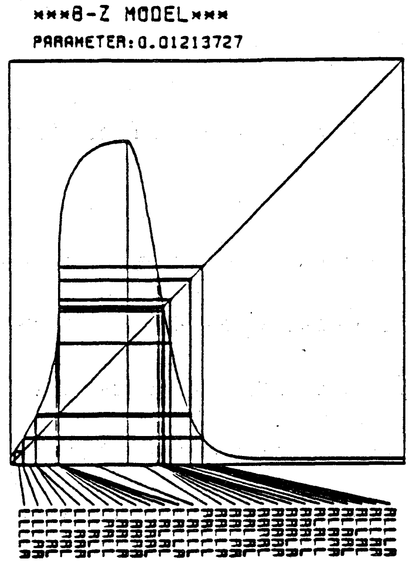

Fig. 4

5th refinement of the markov partition

Modified transition matrix was constructed, the entropy was calculated and was checked with the result of the simulation. It was indicated that the noise can change the dynamics, in other words, there exists the hidden dynamics in this class of maps.

We also studied this phenomenon from a viewpoint of the algorithmic complexity as a digression of previous paper - Noise-induced order -. An interesting aspect was obtained : Noise shortens the computer program. This aspect gives the applicability of this phenomenon to the communication technology. However, the real cause of this phenomenon was left unresolved.

Recently, we studied this problem from the viewpoint of an information flow and solved the real mechanism of this phenomenon. We here discuss the study from this aspect. For the detail see Ref. 4).

2. INFORMATION THEORY OF ONE-DIMENSIONAL MAPPINGS

We measure the informational amount of the probability density p(x) by the Kullback information.

$$I_k(p) = \int p(x) \log\frac{p(x)}{q(x)}\, dx, \tag{2}$$

where q(x) the standard distribution. Usually q is taken as invariant density or uniform density.

The difference of informational amount as the probability distribution changes p(x) to p'(x) can be expressed by the difference of the Kullback information. Particularly in the case of one-dimensional mappings, the difference of the Kullback information between the original and the mapped densities gives the change of informational amount contained in the initial condition. This can be expressed as

$$I_k(p) - I_k(Fp), \tag{3}$$

where F is the Frobenius-Perron operator of a map f.

Taking an appropriate set of densities $\{p_i\}$ which sum up to the invariant density $p_o(x)$, $P_o = \sum_i P_i$, one can obtain the average difference of the Kullback information per iteration by summing the difference for each p_i :

$$\begin{aligned}&(I_k(p_i) - I_k(Fp_i))\\ =\ &\sum_i p_i(x)\int\log\frac{p_i(x)}{q(x)}\, dx - \sum_i \int Fp_i(x) \log\frac{Fp_i(x)}{q(x)}\, dx,\end{aligned} \tag{4}$$

This is called the information flow[5]. Here let us take $p_i(x) = p_o(x)\ \chi_i(x)$, where $\chi_i(x)$ is a characteristic function of one element of an appropriate partition. Then, we have the following expression

$$\sum_i (I_k(P_i) - I_k(FP_i) = \int P_o(x) \sum_i \rho_i(x) \log\frac{1}{\rho_{i(x)}}\, dx, \tag{5}$$

where $\rho_i(x) = \dfrac{Fp_i(x)}{Fp_o(x)}$.

$\rho_i(x)$ is the probability of the previous point of x being in the i-th element of the partition. If the map is one-to-one on each element of the partition, then the right hand side of eq. (5) is independent of the partition. And it is precisely the amount of information required to map a point inversely. Therefore, this quantity is the amount of information of initial condition lost per iteration. This equals to the Lyapounov exponent if $p_o(x)$ is absolutely continuous :

$$\sum_i (I_k(p_i) - I_k(Fp_i)) = \int p_o(x) \log \left|\frac{df}{dx}\right| dx \ (\equiv\lambda).$$

To observe these informational quantities, we introduce the mutual information. One can measure the amount of information transmitted between two places by the mutual information I. Suppose the signal i is sent from one place and at the other place the signal j (not necessarily i) is received. Let p(i,j) be the joint probability of its occurrence and the conditional probability $p(j/i) = \dfrac{p(i,j)}{p(i)}$. Assuming

$P(i) = \sum_j P(i,j) = \sum_j P(j,i)$, the mutual information is defined by

$$I(i;j) = \sum_i p(i) \log \frac{1}{p(i)} - \sum_{ij} p(i)p(j/i) \log \frac{1}{p(j/i)}. \qquad (6)$$

Let us consider the relation between the mutual information and the information flow (4). Devide the unit interval into, say 100 equal segments and let $p_o(i)$ be the probability of finding the orbital point in the i-th segment. Then $p_o(j/i)$ is the transition probability under the mapping from i to j. We obtain approximately the expression $F\chi_i(x) \cong p_o(j/i)\,\chi_j(x)$. Substituting this expression into eq. (5), we obtain

$$(I_k(p_i) - I_k(Fp_i)) \cong \sum_{ij} p(i)p(j/i) \log \frac{1}{p(j/i)} \qquad (7)$$

Therefore, the second term of the right hand side of eq. (6) is just the expression (7). If the partition becomes finer, the first term of the right hand side of eq. (6) becomes larger, but the second term remains <u>nearly</u> equal to the information flow or the information loss (eq. (5)).

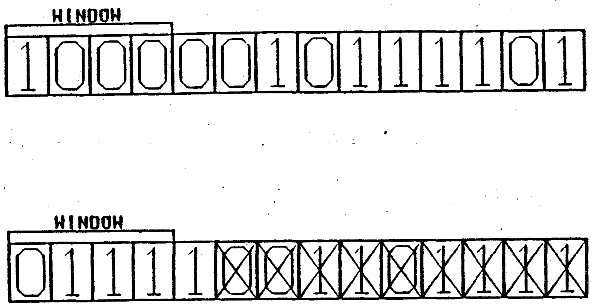

Fig.5

Computer register : the above figure denotes the noiseless case and the below the noisy case.

It is convenient to consider the "computer register" (Fig. 5) in order to explain the difference between the mutual information and the information flow. The above partition corresponds to observing this register only through highest, say four places. The mutual informatin accounts for every information escaping from this window. But, the information flow has a sigh according to its direction and only its average is matter of concern. The information loss is equal to the information flow. Only in the case when the fluctuation of information flow is small, we obtain the equality in the expression (7). In the other words, one can see the fluctuation of information flow in terms of the mutual information.

3. RESULT OF THE MUTUAL INFORMATION

We calculate the mutual information by deviding the unit interval into 100 equal segments, and using next expression,

$$I_n(i;j) = \sum_i p_o(i) \log \frac{1}{p_o(i)} - \sum_{ij} p_o(i)p_n(j/i) \log \frac{1}{p_n(j/i)} , \qquad (8)$$

where $p_n(j/i)$ is the conditional probability of a point started from the i-th segment falling in the j-th segment after n iterations. We calculate $I_n(i;j)$ for the B-Z map and the logistic map with and without noise. The results are shown in Fig. 6a)-d). We discuss the following three features of the mutual information curve.

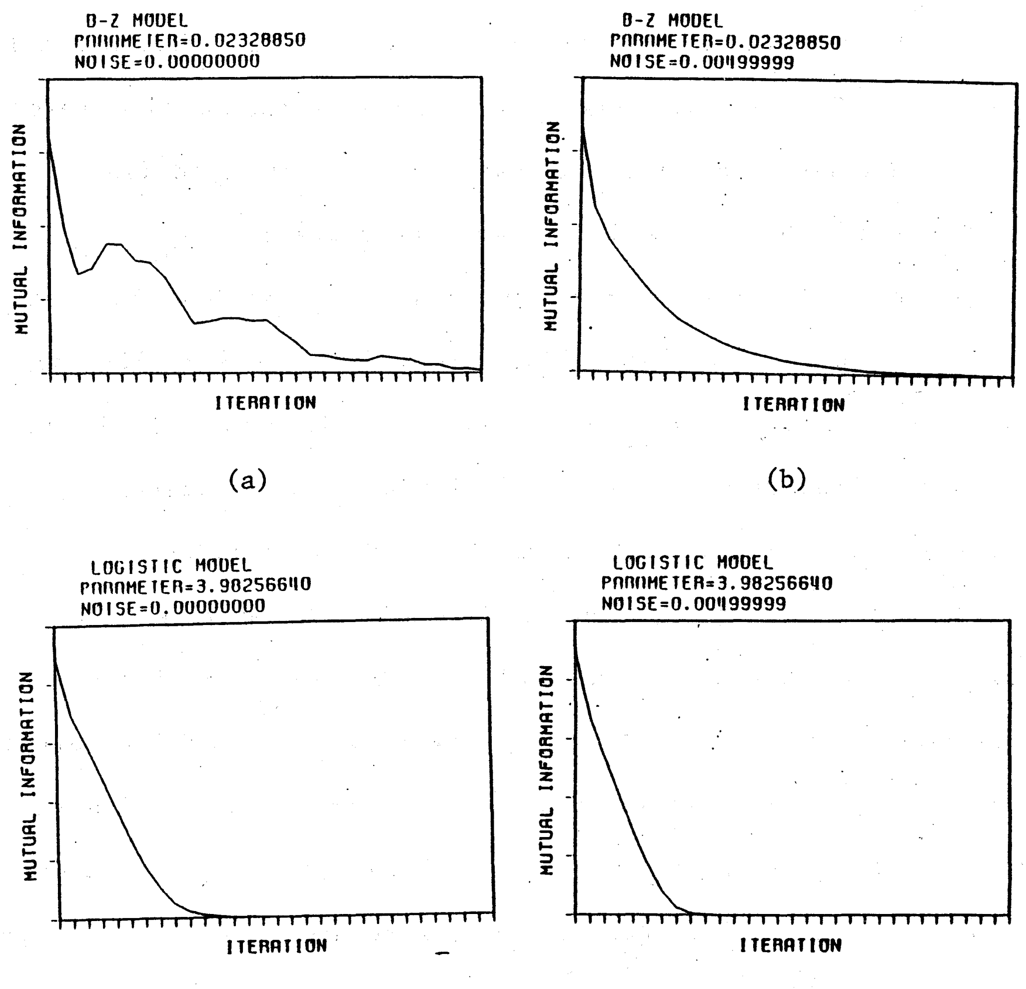

Fig. 6 Mutual information : (a) and (b) are noiseless case and noisy case respectively for the B-Z map. (c) and (d) are noiseless case and noisy case respectively for the logistic map

(i) the linear decrease in the logistic map,

(ii) the exponential decrease in the B-Z map,

(iii) the humps in the noise-free B-Z map.

The linear decrease corresponds to a simple picture of the information flow, namely the average information flow consists of the monotonic flow directed to the left had side of the register in Fig. 5. Therefore, the mutual information loses as much as the information flow carries away. Even in the noisy case, the information flow never be caught by the noise, since there is not any right directed flow in this case (uniform attractor case). We have the expression $I_n = I_o - \lambda n$, where λ is the Lyapounov exponent, since the information flow rate is equal to the Lyapounov exponent. This situation is essentially the same as in the Bernoulli map ($X_{n+1} = 2X_n$(Mod 1)). Namely, this is in the case that the fluctuation of the information flow is small. (In the case of the Bernoulli map, the information flow never fluctuate.)

In the case of the exponential decrease, the average information flow directed to the left hand side of the register is a sum of contributions from the right directed flow as well as the left directed flow. So, this is in the case of the large fluctuation of the information flow, which produces the breakdown of eq. (7). Generally, in the noiseless case, the information loss occurs at the largest scale (i.e., at the left end of the register) as seen in eq. (5). The information flowed to the right hand side of the register must return to the window since the information loss mechanism does not exist there. This gives the reason of the appearance of the hump.

But, in the noisy case, the additional information loss michanism works near the right end of the register. The information carried by the right directed flow is caught by the noise and perished.

So we have the information mixing from which the mutual information decreases at the same rate in any time. Namely, $I_n = I_o r^n$. As the information loss equals to the Lyapounov exponent, $I_o(1-r) = \lambda$ holds. This condition clearly gives the real mechanism of the noise-induced order. The information mixing is responsible for the drastic change of the dynamics.

4. SUMMARY AND OUTLOOK

We obtained two types of decay of the information about the initial condition. One is a linear decay and the other is an exponential decay. In the former case, the initially localized information remains localized in subsequent time. In the latter case, the initially localized information spreads considerably and an information mixing occurs. Noise-induced order corresponds to the latter case, where the chaos itself is fragile, since extended information is easily destroyed by noise.

One cannot overlook the important application of the present result to the memory mechanism of the brain. The exponential decay of the information obtained here clearly indicates that a part of the whole information are contained in each place of the register. This shows the possibility to give a concrete mechanism of the holographic memory of the brain. In the brain, the maps of the B-Z type (not of the logistic type) might exist.

REFERENCES

1) K. Matsumoto and I. Tsuda, J. Stat. Phys. 31 : 87 (1983)

2) K. Matsumoto, J. Stat. Phys. 34 : 111 (1983)

3) I. Tsuda and K. Matsumoto, in "Chaos and Statistical Methods" ed) Y. Kuramoto, p 102 (1984) - Springer Series in Synergetics -

4) K. Matsumoto and I. Tsuda, to appear in J. Stat. Phys.

5) R. Shaw, Zeit. fur Naturforschung 36a : 80 (1981)